

Generalized energies and integrable $D_n^{(1)}$ cellular automaton

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Dedicated to Professor Tetsuji Miwa on his 60th birthday

ABSTRACT: We introduce generalized energies for a class of $U_q(D_n^{(1)})$ crystals by using the piecewise linear functions that are building blocks of the combinatorial R . They include the conventional energy in the theory of affine crystals as a special case. It is shown that the generalized energies count the particles and anti-particles in a quadrant of the two dimensional lattice generated by time evolutions of an integrable $D_n^{(1)}$ cellular automaton. Explicit formulas are conjectured for some of them in the form of ultradiscrete tau functions.

1. INTRODUCTION

Let B_l be the crystal of the l -fold symmetric tensor representation of the quantum affine algebra $U_q(D_n^{(1)})$ [11, 9]. The combinatorial $R : x \otimes y \mapsto y' \otimes x'$ is the isomorphism of crystals $B_l \otimes B_m \xrightarrow{\sim} B_m \otimes B_l$ corresponding to the quantum R at $q = 0$ [10]. In Ref. [15], an explicit formula of the combinatorial R was obtained in terms of several piecewise linear functions $g_i(x \otimes y) \in \mathbb{Z}$ on $B_l \otimes B_m$. See Theorem 2.1. Among them is the *local energy*, which plays an essential role in the theory of affine crystals [10]. The family of piecewise linear functions $\{g_i\}$, which we call *generalized local energies* in this paper, are ultradiscretization of the subtraction-free rational functions that have emerged as building blocks of the tropical R [15, 16] of the geometric crystal [3]. They may be viewed as local energies in a principal picture rather than in the conventional homogeneous picture.

From the local energy, one can form the integer-valued function called *energy* on the tensor product $\mathcal{P} = B_{l_1} \otimes \cdots \otimes B_{l_L}$. Its generating function is the one dimensional configuration sum that originates in the corner transfer matrix method [1, 2].

In this paper we introduce *generalized energies* $\mathcal{E}_{g_i} : \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}$ corresponding to g_i 's, and study them from the viewpoint of the integrable cellular automaton of type $D_n^{(1)}$ [6, 7]. The latter is an integrable $U_q(D_n^{(1)})$ vertex model at $q = 0$. It is a dynamical system on \mathcal{P} equipped with commuting time evolutions $\{T_l\}_{l \geq 1}$. Elements of \mathcal{P} are naturally regarded as arrays of particles and anti-particles, and T_l induces their factorized scattering involving pair creation and annihilation. See Examples 3.1 and 3.2.

Our main result is Theorem 4.1, which states that $\mathcal{E}_{g_i}(p) = \rho_{g_i}(p)$ for any $p = p_1 \otimes \cdots \otimes p_L \in \mathcal{P}$. Here $\rho_{g_i}(p)$ is a *counting function* giving the number of certain particles and anti-particles specified by g_i in the region (3.3) under the time evolutions $p, T_\infty(p), T_\infty^2(p), \dots$. As such, the counting functions are non-local variables attached to a quadrant of the 2 dimensional lattice. However, it will also be shown in Theorem 3.5 that the combined data $\{\rho_{g_i}(p_1 \otimes \cdots \otimes p_j) \mid j = k-1, k\}$ in turn reproduces the local variable $p_k \in B_{l_k}$ completely in agreement with the spirit of the corner transfer matrix method. Therefore the joint spectrum $\{\mathcal{E}_{g_i}(p_1 \otimes \cdots \otimes p_k)\}$ of the generalized energies with $1 \leq k \leq L$ is equivalent to $p = p_1 \otimes \cdots \otimes p_L \in \mathcal{P}$ itself. This extends a similar result on type $A_n^{(1)}$ (Proposition 4.6 in Ref. [17]) which is related to the katabolism [22]. A supplementary result (Proposition 4.3) is parallel with Theorem 4.1 and treats generalized local energies with opposite chirality (cf. Remark 2.5).

The layout of the paper is as follows. In Section 2, generalized (local) energies are extracted from the piecewise linear formula of the combinatorial R [15]. In Section 3, the integrable $D_n^{(1)}$ cellular automaton [6, 7] is recalled and the counting functions are defined. In Section 4, the main Theorem 4.1 of the paper is stated and proved. In Section 5, aspects related to combinatorial Bethe ansatz are discussed. In Section 5.1 we give

the inverse scattering formalism of the $D_n^{(1)}$ cellular automaton like Ref. [14]. In Section 5.2, we conjecture piecewise linear formulas for some generalized energies in terms of *ultradiscrete tau functions*. This is also motivated by the $A_n^{(1)}$ case [17], where analogous results have led to a piecewise linear formula for the Kerov-Kirillov-Reshetikhin map [13]. Although the conjecture is yet to cover the full family of generalized energies, the last one (5.8) is already rather intriguing. We expect that the extension and the solution of Conjecture 5.3 will uncover an interplay among combinatorial Bethe ansatz, ultradiscretization of the DKP hierarchy [8] and the bilinearization of the tropical R [16].

2. GENERALIZED ENERGIES FOR $D_n^{(1)}$ CRYSTAL

2.1. Crystals and combinatorial R . Let us recall the basic facts on crystal and combinatorial R briefly. For a more information, see Refs. [10, 11, 9] and [19]. For a positive integer l , let

$$(2.1) \quad B_l = \{\zeta = (\zeta_1, \dots, \zeta_n, \bar{\zeta}_n, \dots, \bar{\zeta}_1) \in \mathbb{Z}_{\geq 0}^{2n} \mid \sum_{i=1}^n (\zeta_i + \bar{\zeta}_i) = l, \zeta_n \bar{\zeta}_n = 0\}$$

be the crystal of the l -fold symmetric tensor representation of $U_q(D_n^{(1)})$ [9]. We assume $n \geq 3$. As for the functions ε_i, φ_i , the tensor product rule and the action of Kashiwara operators \tilde{e}_i and \tilde{f}_i ($0 \leq i \leq n$), see Ref. [19].

The *affinization* of the crystal B_l is defined by $\text{Aff}(B_l) = \{b[d] \mid d \in \mathbb{Z}, b \in B_l\}$ with the crystal structure $\tilde{e}_i(b[d]) = (\tilde{e}_i b)[d + \delta_{i0}]$ and $\tilde{f}_i(b[d]) = (\tilde{f}_i b)[d - \delta_{i0}]$. We call b and d the classical and the affine part of $b[d]$, respectively. There exists the unique bijection (crystal isomorphism) $B_l \otimes B_m \xrightarrow{\sim} B_m \otimes B_l$ that commutes with all Kashiwara operators. It is lifted up to a map $\text{Aff}(B_l) \otimes \text{Aff}(B_m) \xrightarrow{\sim} \text{Aff}(B_m) \otimes \text{Aff}(B_l)$ called the *combinatorial R* , which has the following form:

$$\begin{aligned} R : \text{Aff}(B_l) \otimes \text{Aff}(B_m) &\longrightarrow \text{Aff}(B_m) \otimes \text{Aff}(B_l) \\ b[d] \otimes b'[d'] &\longmapsto \tilde{b}'[d' + H(b \otimes b')] \otimes \tilde{b}[d - H(b \otimes b')], \end{aligned}$$

where $b \otimes b' \mapsto \tilde{b}' \otimes \tilde{b}$ under the isomorphism $B_l \otimes B_m \xrightarrow{\sim} B_m \otimes B_l$.¹ The quantity $H(b \otimes b')$ is called the *local energy* and determined up to a global additive constant by

$$H(\tilde{e}_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0, \varphi_0(b) \geq \varepsilon_0(b'), \varphi_0(\tilde{b}') \geq \varepsilon_0(\tilde{b}), \\ H(b \otimes b') - 1 & \text{if } i = 0, \varphi_0(b) < \varepsilon_0(b'), \varphi_0(\tilde{b}') < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise.} \end{cases}$$

The Yang-Baxter equation

$$(2.2) \quad (R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$$

is satisfied on $\text{Aff}(B_l) \otimes \text{Aff}(B_m) \otimes \text{Aff}(B_k)$.

2.2. Generalized local energies. Let us give an explicit piecewise linear formula of the combinatorial R that originates in the tropical R for geometric crystals of type $D_n^{(1)}$ [15]. First we make a slight variable change. The set B_l (2.1) is in one to one correspondence with another set

$$(2.3) \quad \begin{aligned} B'_l &= \{x = (x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) \in \mathbb{Z}^{2n-1} \mid x_i, \bar{x}_i \geq 0 \text{ for } 1 \leq i \leq n-1, \\ &\quad x_n \geq -\min(x_{n-1}, \bar{x}_{n-1}), \sum_{i=1}^{n-1} (x_i + \bar{x}_i) + x_n = l\} \end{aligned}$$

¹ This classical part of the combinatorial R will also be referred as combinatorial R and denoted by $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$.

by the relations

$$(2.4) \quad x_i = \zeta_i, \quad \bar{x}_i = \bar{\zeta}_i \quad (1 \leq i \leq n-2),$$

$$(2.5) \quad x_{n-1} = \zeta_{n-1} + \bar{\zeta}_n, \quad x_n = \zeta_n - \bar{\zeta}_n, \quad \bar{x}_{n-1} = \bar{\zeta}_{n-1} + \bar{\zeta}_n,$$

$$(2.6) \quad \zeta_n = \max(0, x_n), \quad \bar{\zeta}_n = \max(0, -x_n),$$

$$(2.7) \quad \zeta_{n-1} = x_{n-1} + \min(0, x_n), \quad \bar{\zeta}_{n-1} = \bar{x}_{n-1} + \min(0, x_n).$$

Note that x_n can be negative. We naturally use the notations like $x[d] \in \text{Aff}(B'_l)$ and $R(x[d] \otimes x'[d']) = \tilde{x}'[d' + H(x \otimes x')] \otimes \tilde{x}[d - H(x \otimes x')]$, etc. Set

$$(2.8) \quad \ell(\zeta) = \sum_{i=1}^n (\zeta_i + \bar{\zeta}_i) \quad (\zeta \in B_l), \quad \ell(x) = \sum_{i=1}^{n-1} (x_i + \bar{x}_i) + x_n \quad (x \in B'_l),$$

so that $\ell(\zeta) = \ell(x) = l$ for $\zeta \in B_l$ and $x \in B'_l$.

Let $x = (x_1, \dots, \bar{x}_1) \in B'_l$ and $y = (y_1, \dots, \bar{y}_1) \in B'_m$. On the pair (x, y) we introduce mutually commuting involutions σ_1, σ_n and $*$ by

$$(2.9) \quad (x, y)^{\sigma_1} = (x^{\sigma_1}, y^{\sigma_1}), \quad (x, y)^{\sigma_n} = (x^{\sigma_n}, y^{\sigma_n}), \quad (x, y)^* = (y^*, x^*),$$

$$\sigma_1 : x_1 \longleftrightarrow \bar{x}_1,$$

$$\sigma_n : x_{n-1} \rightarrow x_{n-1} + x_n, \quad \bar{x}_{n-1} \rightarrow \bar{x}_{n-1} + x_n, \quad x_n \rightarrow -x_n,$$

$$(2.10) \quad * : x_i \longleftrightarrow \bar{x}_i \quad (1 \leq i \leq n-1).$$

The coordinates not included in the above rules are left unchanged. These involutions are naturally defined on $(\xi, \zeta) \in B_l \times B_m$ as well by the correspondence (2.4)-(2.7). For instance, one has $(\xi, \zeta)^* = (\zeta^*, \xi^*)$ with $\zeta^* = (\bar{\zeta}_1, \dots, \bar{\zeta}_{n-1}, \zeta_n, \bar{\zeta}_n, \zeta_{n-1}, \dots, \zeta_1)$ for $\zeta = (\zeta_1, \dots, \zeta_n, \bar{\zeta}_n, \dots, \bar{\zeta}_1)$.

For any function $g = g(x, y)$, we write $g^{\sigma_1} = g^{\sigma_1}(x, y) = g(x^{\sigma_1}, y^{\sigma_1})$, etc. Introduce the piecewise linear functions $V_i = V_i(x, y)$ and $W_i = W_i(x, y)$ for $0 \leq i \leq n-1$ as follows.

$$(2.11) \quad V_i = \max \left(\{\theta_{i,j}, \theta'_{i,j} | 1 \leq j \leq n-2\} \cup \{\eta_{i,j}, \eta'_{i,j} | 1 \leq j \leq n\} \right),$$

$$(2.12) \quad W_0 = 2V_0, \quad W_1 = V_0 + V_0^{\sigma_1}, \quad W_{n-1} = V_{n-1} + V_{n-1}^*,$$

$$(2.13) \quad W_i = \max \left(V_i + V_{i-1}^* - y_i, V_{i-1} + V_i^* - \bar{x}_i \right) + \min(x_i, \bar{y}_i),$$

where $2 \leq i \leq n-2$ in the last line. The functions $\theta_{i,j} = \theta_{i,j}(x, y)$, $\theta'_{i,j} = \theta'_{i,j}(x, y)$, $\eta_{i,j} = \eta_{i,j}(x, y)$, $\eta'_{i,j} = \eta'_{i,j}(x, y)$ are defined by

$$\theta_{i,j}(x, y) = \begin{cases} \ell(x) + \sum_{k=j+1}^i (\bar{y}_k - \bar{x}_k) & \text{for } 1 \leq j \leq i, \\ \ell(y) + \sum_{k=i+1}^j (\bar{x}_k - \bar{y}_k) & \text{for } i+1 \leq j \leq n-2, \end{cases}$$

$$\theta'_{i,j}(x, y) = \ell(x) + \sum_{k=1}^i (\bar{y}_k - \bar{x}_k) + \sum_{k=1}^j (y_k - x_k) \quad \text{for } 1 \leq j \leq n-2,$$

TABLE 1. Transformation by $\sigma_1, \sigma_n, *$ and R .

	V_0	$V_i (1 \leq i \leq n-2)$	V_{n-1}	$W_i (1 \leq i \leq n-1)$
σ_1	$V_0^{\sigma_1}$	V_i	V_{n-1}	W_i
σ_n	V_0	V_i	V_{n-1}^*	W_i
$*$	V_0	V_i^*	V_{n-1}^*	W_i
R	V_0	$W_i - V_i^*$	V_{n-1}	W_i

$$\eta_{i,j}(x, y) = \begin{cases} \ell(x) + \sum_{k=j+1}^i (\bar{y}_k - \bar{x}_k) + \bar{y}_j - x_j & \text{for } 1 \leq j \leq i, \\ \ell(y) + \sum_{k=i+1}^j (\bar{x}_k - \bar{y}_k) + \bar{y}_j - x_j & \text{for } i+1 \leq j \leq n-1, \\ \ell(y) + \sum_{k=i+1}^{n-1} (\bar{x}_k - \bar{y}_k) + x_n & \text{for } j = n, \end{cases}$$

$$\eta'_{i,j}(x, y) = \begin{cases} \ell(x) + \sum_{k=1}^i (\bar{y}_k - \bar{x}_k) + \sum_{k=1}^j (y_k - x_k) + x_j - \bar{y}_j & \text{for } 1 \leq j \leq n-1, \\ \ell(x) + \delta_{i,n-1} (\ell(x) - \ell(y)) + \sum_{k=1}^i (\bar{y}_k - \bar{x}_k) + \sum_{k=1}^{n-1} (y_k - x_k) - x_n & \text{for } j = n. \end{cases}$$

Theorem 2.1 (Ref. [15], Theorem 4.28 and Remark 4.29). *The image $y' \otimes x' = R(x \otimes y)$ of the combinatorial R is given by*

$$(2.14) \quad \begin{aligned} x'_i &= x_i + V_{i-1}^* - V_i^*, & \bar{x}'_i &= \bar{x}_i + V_{i-1}^* + W_i - V_i^* - W_{i-1} \quad (1 \leq i \leq n-1), \\ x'_n &= x_n + V_{n-1}^* - V_{n-1}, & y'_n &= y_n + V_{n-1} - V_{n-1}^*, \\ y'_i &= y_i + V_{i-1} + W_i - V_i - W_{i-1}, & \bar{y}'_i &= \bar{y}_i + V_{i-1} - V_i \quad (1 \leq i \leq n-1). \end{aligned}$$

Moreover, the local energy is given by

$$(2.15) \quad H(x \otimes y) = V_0(x, y)$$

up to a constant shift.

The functions $V_1, \dots, V_{n-1}, W_1, \dots, W_{n-1}$ and σ_1, σ_n and $*$ of them are relatives of the local energy. In addition to the involutions σ_1, σ_n and $*$, the combinatorial R naturally acts on them by $(RV_0)(x, y) = V_0(y', x')$ with $y' \otimes x' = R(x \otimes y)$, etc. Their transformation properties under $\sigma_1, \sigma_n, *$ and R are summarized in Table 1 [15]. These involutions are commutative, thus for instance $R(V_0^{\sigma_1}) = (R(V_0))^{\sigma_1} = V_0^{\sigma_1}$.

Due to these properties, there are a few simplifications in (2.14) as

$$(2.16) \quad \begin{aligned} x'_1 &= x_1 + V_0 - V_1^*, & \bar{x}'_1 &= \bar{x}_1 + V_0^{\sigma_1} - V_1^*, \\ y'_1 &= y_1 + V_0^{\sigma_1} - V_1, & \bar{y}'_1 &= \bar{y}_1 + V_0 - V_1. \end{aligned}$$

We write

$$(2.17) \quad u_l = (l, 0, \dots, 0) \in B_l.$$

By using Theorem 2.1, one can show for any $\zeta \in B_m$ that

$$(2.18) \quad B_l \otimes B_m \ni u_l \otimes \zeta \xrightarrow{\sim} u_m \otimes \xi' \in B_m \otimes B_l \text{ if } l \geq m$$

for some ξ' under the combinatorial R . In particular

$$u_l \otimes u_m \simeq u_m \otimes u_l$$

holds. The functions in Table 1 attain their maximum $V_0 = V_0^{\sigma_1} = V_i = V_i^* = l + m$ and $W_i = 2(l + m)$ for $1 \leq i \leq n - 1$ at $(x, y) = (u_l, u_m)$.

For $\xi \otimes \zeta \in B_l \otimes B_m$, let $x \in B'_l$ and $y \in B'_m$ be the elements corresponding to ξ and ζ , respectively. We set

$$(2.19) \quad v_i(\xi \otimes \zeta) = \ell(\xi) + \ell(\zeta) - V_i(x, y) \quad (0 \leq i \leq n - 1)$$

$$(2.20) \quad v_0^{\sigma_1}(\xi \otimes \zeta) = \ell(\xi) + \ell(\zeta) - V_0^{\sigma_1}(x, y),$$

$$(2.21) \quad v_i^*(\xi \otimes \zeta) = \ell(\xi) + \ell(\zeta) - V_i^*(x, y) \quad (1 \leq i \leq n - 1),$$

$$(2.22) \quad w_i(\xi \otimes \zeta) = 2\ell(\xi) + 2\ell(\zeta) - W_i(x, y) \quad (1 \leq i \leq n - 1),$$

and call them *generalized local energies*. Note that $w_{n-1} - v_{n-1} = v_{n-1}^*$. They are building blocks of the piecewise linear formula of the combinatorial R (2.14). From the above remark, generalized local energies are all nonnegative and normalized so that

$$(2.23) \quad g(u_l \otimes u_m) = 0 \quad \text{for any } g = v_i, v_0^{\sigma_1}, v_i^* \text{ and } w_i.$$

For $\zeta = (\zeta_1, \dots, \zeta_n, \bar{\zeta}_n, \dots, \bar{\zeta}_1) \in B_l$, we introduce

$$(2.24) \quad \mathbf{a}(\zeta) = \zeta_2 + \dots + \zeta_n + \bar{\zeta}_n + \dots + \bar{\zeta}_2 + 2\bar{\zeta}_1 = \ell(\zeta) + \bar{\zeta}_1 - \zeta_1,$$

$$(2.25) \quad \gamma_{v_a}(\zeta) = \zeta_2 + \dots + \zeta_n + \bar{\zeta}_n + \dots + \bar{\zeta}_{a+1} + \bar{\zeta}_1 \quad (0 \leq a \leq n - 2),$$

$$(2.26) \quad \gamma_{v_{n-1}}(\zeta) = \zeta_2 + \dots + \zeta_{n-1} + \zeta_n + \bar{\zeta}_1,$$

$$(2.27) \quad \gamma_{v_{n-1}^*}(\zeta) = \zeta_2 + \dots + \zeta_{n-1} + \bar{\zeta}_n + \bar{\zeta}_1,$$

$$(2.28) \quad \gamma_{w_a - v_a}(\zeta) = \zeta_2 + \dots + \zeta_a + \bar{\zeta}_1 \quad (1 \leq a \leq n - 2),$$

$$(2.29) \quad \gamma_{v_0^{\sigma_1}}(\zeta) = 0.$$

Note that $\gamma_{v_0}(\zeta) = \mathbf{a}(\zeta)$.

Lemma 2.2. *Let $\xi = (\xi_1, \dots, \bar{\xi}_1) \in B_l$ and $\zeta = (\zeta_1, \dots, \bar{\zeta}_1) \in B_m$. Set $\zeta' \otimes \xi' = R(\xi \otimes \zeta) \in B_m \otimes B_l$. For ξ_1 (hence l as well) sufficiently large, the relation*

$$g(\xi \otimes \zeta) = \gamma_g(\zeta) + \mathbf{a}(\zeta') - \gamma_g(\zeta')$$

is valid for g appearing in (2.25)–(2.29), where the left hand side with $g = w_a - v_a$ is to be understood as $w_a(\xi \otimes \zeta) - v_a(\xi \otimes \zeta)$.

Proof. One can check that $\xi_1 \geq m - \zeta_1 + \bar{\zeta}_1$ is sufficient to guarantee that V_0 (2.11) is equal to $\eta'_{0,1} = l + \zeta_1 - \bar{\zeta}_1$. This implies that $v_0(\xi \otimes \zeta) = m - \zeta_1 + \bar{\zeta}_1 = \gamma_{v_0}(\zeta)$ showing the $g = v_0$ case. All the other cases are deduced from this, (2.14) and (2.4)–(2.7) without using the concrete forms of $\theta_{i,j}, \theta'_{i,j}, \eta_{i,j}$ and $\eta'_{i,j}$. \square

For $g = w_a$ ($1 \leq a \leq n - 2$), an analogue of Lemma 2.2 holds with

$$(2.30) \quad \begin{aligned} w_a(\xi \otimes \zeta) &= \gamma_{w_a}(\zeta) + 2\mathbf{a}(\zeta') - \gamma_{w_a}(\zeta'), \\ \gamma_{w_a}(\zeta) &= \gamma_{w_a - v_a}(\zeta) + \gamma_{v_a}(\zeta) \\ &= \mathbf{a}(\zeta) + \zeta_2 + \dots + \zeta_a - (\bar{\zeta}_a + \dots + \bar{\zeta}_2). \end{aligned}$$

As ξ_1 gets large, ζ' stabilizes since it is a piecewise linear function of ξ_1 staying in a finite set B_m . ($\xi_1 \geq m$ seems sufficient for the convergence.) Therefore Lemma 2.2 ensures that all the generalized local energies $g(\xi \otimes \zeta)$ are well defined in the limit $\xi_1 \rightarrow \infty$.

2.3. Generalized energies. For $p = p_1 \otimes \dots \otimes p_L \in B_{l_1} \otimes \dots \otimes B_{l_L}$, define $p_j^{(i)} \in B_{l_j}$ ($i < j$) by

$$(2.31) \quad \begin{aligned} (B_{l_i} \otimes \dots \otimes B_{l_{j-1}}) \otimes B_{l_j} &\xrightarrow{\sim} B_{l_j} \otimes (B_{l_i} \otimes \dots \otimes B_{l_{j-1}}) \\ p_i \otimes \dots \otimes p_{j-1} \otimes p_j &\mapsto p_j^{(i)} \otimes p'_i \otimes \dots \otimes p'_{j-1}, \end{aligned}$$

sending p_j to the left by successive applications of the combinatorial R . We set $p_j^{(j)} = p_j$. For any generalized local energy in (2.19)–(2.22), we define the *generalized energy* of $p = p_1 \otimes \cdots \otimes p_L \in B_{l_1} \otimes \cdots \otimes B_{l_L}$ by

$$(2.32) \quad \mathcal{E}_g(p) = \sum_{0 \leq i < j \leq L} g(p_i \otimes p_j^{(i+1)})$$

by taking $p_0 = u_l$ with sufficiently large l . This is well defined (finite) due to Lemma 2.2 and the comment following it. In the rest of the paper, we will simply write $p_0 = u_\infty \in B_\infty$.

When $g = v_0$, (2.32) is the energy introduced in Refs. [18] and [5] up to a sign and a constant shift. If furthermore l_1, \dots, l_L are all equal, then $p_j^{(i+1)} = p_{i+1}$ holds and (2.32) reduces to

$$\mathcal{E}_{v_0}(p) = \sum_{0 \leq i < L} (L - i) v_0(p_i \otimes p_{i+1}).$$

Its generating function $\sum_p q^{\mathcal{E}_{v_0}(p)}$ is a version of the one dimensional configuration sum going back to Refs. [2] and [1], which is the essential ingredient in the corner transfer matrix method.

Any quantity $G(p_\alpha \otimes p_{\alpha+1} \otimes \cdots \otimes p_\beta)$ will be said *R-invariant* if $G(\cdots \otimes p_i \otimes p_{i+1} \otimes \cdots) = G(\cdots \otimes R(p_i \otimes p_{i+1}) \otimes \cdots)$ for any $\alpha \leq i < \beta$.

Remark 2.3. Due to the transformation property under R in Table 1, the generalized energy $\mathcal{E}_g(p)$ is *R-invariant* for $g = v_0, v_0^{\sigma_1}, v_{n-1}, v_{n-1}^*$ and w_1, \dots, w_{n-1} . On the other hand, \mathcal{E}_g with $g = v_1, \dots, v_{n-2}$ and v_1^*, \dots, v_{n-2}^* are *not R-invariant*.

Let us depict the relation $R(b \otimes c) = \tilde{c} \otimes \tilde{b}$ as

$$\begin{array}{cc} b & c \\ & \times \\ \tilde{c} & \tilde{b} \end{array}$$

Then the Yang-Baxter equation (2.2) takes the well known form:

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$$

The defining relation (2.31) of $p_j^{(i)}$ looks as

$$(2.33) \quad \begin{array}{c} p_{j-1} \quad p_j \\ \diagdown \quad \diagup \\ p_{j-2} \quad \diagdown \quad \diagup \\ \vdots \quad \diagdown \quad \diagup \\ p_i \quad \diagdown \quad \diagup \\ p_j^{(i)} \end{array}$$

Remember that each vertex is associated with various generalized local energies $g(b \otimes c)$. Let

$$(2.34) \quad I_g = I_g(p_i \otimes \cdots \otimes p_{j-1} \otimes p_j) = \sum_{i \leq k < j} g(p_k \otimes p_j^{(k+1)})$$

be the sum of generalized local energy g over all the vertices in (2.33). Then the generalized energy (2.32) is expressed as

$$(2.35) \quad \begin{aligned} \mathcal{E}_g(p_1 \otimes \cdots \otimes p_L) &= \mathcal{E}_g(p_1 \otimes \cdots \otimes p_{L-1}) + I_g(u_\infty \otimes p_1 \otimes \cdots \otimes p_L) \\ &= \sum_{1 \leq j \leq L} I_g(u_\infty \otimes p_1 \otimes \cdots \otimes p_{j-1} \otimes p_j). \end{aligned}$$

From (2.18) and (2.23), it follows that

$$(2.36) \quad I_g(u_\infty \otimes u_\infty \otimes p_1 \otimes \cdots \otimes p_j) = I_g(u_\infty \otimes p_1 \otimes \cdots \otimes p_j),$$

$$(2.37) \quad \mathcal{E}_g(u_\infty \otimes p_1 \otimes \cdots \otimes p_L) = \mathcal{E}_g(p_1 \otimes \cdots \otimes p_L).$$

Lemma 2.4. *In (2.33), the following quantities are R -invariant as the functions of $p_i \otimes \cdots \otimes p_{j-1}$. (i) The element $p_j^{(i)}$. (ii) I_g (2.34) for any $g = v_a$ ($0 \leq a \leq n-1$), $w_a - v_a$ ($1 \leq a \leq n-2$), v_{n-1}^* and $v_0^{\sigma_1}$.*

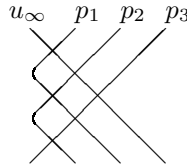
Proof. (i) This is due to the classical part of the Yang-Baxter equation (2.2). (ii) The R -invariance of I_{v_0} follows from (2.19), (2.15) and the affine part of the Yang-Baxter equation. Let $y, y' \in B'_{l_j}$ be the elements corresponding to $p_j, p_j^{(i)} \in B_{l_j}$, respectively. From (2.16), we have $\overline{y}'_1 - \overline{y}_1 = I_{v_1} - I_{v_0}$. Since the left hand side is R -invariant by (i), this relation implies the R -invariance of I_{v_1} . By similarly using the R -invariance of $\overline{y}'_i - \overline{y}_i$ and $y'_i - y_i$ in (2.16) and (2.14), one can verify the R -invariance of the other I_g . \square

We note that Lemma 2.4 is applicable to the situation $i = 0$, i.e., $p_i \otimes \cdots \otimes p_{j-1} = u_\infty \otimes p_1 \otimes \cdots \otimes p_{j-1}$.

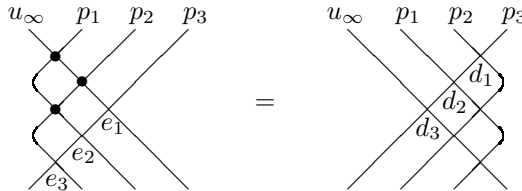
Remark 2.5. Lemma 2.4 (ii) does not concern $I_{v_1^*}, \dots, I_{v_{n-1}^*}$. In fact the proof does not persist since V_1^*, \dots, V_{n-2}^* are not contained in any difference of the components of y' and y in (2.14). Similarly, V_1, \dots, V_{n-2} do not appear in the differences of x and x' . This *chirality* of the combinatorial R is a characteristic feature of the $D_n^{(1)}$ case. In contrast, I_g 's with all the generalized local energies g for $A_n^{(1)}$ (so called i th (un)winding number [17]) are R -invariant. We shall come back to this point again in Section 4.2.

The following proposition and its proof are parallel with Lemma 4.4 in Ref. [17] for type $A_n^{(1)}$.

Proposition 2.6. *For $g = v_i$ ($0 \leq i \leq n-1$), w_i ($1 \leq i \leq n-1$), $v_0^{\sigma_1}$ and v_{n-1}^* , the generalized energy $\mathcal{E}_g(p_1 \otimes \cdots \otimes p_L)$ (2.32) is equal to the sum of the generalized local energy g attached to all the vertices in the following diagram ($L = 3$ example):*



Proof. We invoke the induction on L . For $L = 1$, one has $\mathcal{E}_g(p_1) = g(u_\infty \otimes p_1)$, and the assertion is obviously true. We illustrate the induction step from $L = 2$ to $L = 3$. Consider the following identity obtained by successive applications of the Yang-Baxter equation:



Here \bullet, e_i, d_i stand for the values of g at the attached vertices. By the induction assumption, the sum of the three \bullet in the left hand side is equal to $\mathcal{E}_g(p_1 \otimes p_2)$. In view of the recursion relation (2.35), we are to verify $e_1 + e_2 + e_3 = I_g(u_\infty \otimes p_1 \otimes p_2 \otimes p_3)$. By the definition, $I_g(u_\infty \otimes p_1 \otimes p_2 \otimes p_3) = d_1 + d_2 + d_3$ in the right diagram, where $d_1 = g(p_2 \otimes p_3^{(3)})$, $d_2 = g(p_1 \otimes p_3^{(2)})$, $d_3 = g(u_\infty \otimes p_3^{(1)})$. Thanks to the R -invariance of I_g in Lemma 2.4 (ii), this is equal to $e_1 + e_2 + e_3$. \square

For l_1, \dots, l_L general, T_∞ still admits a similar, although slightly more involved, algorithm. We omit it here and give an example instead.

Example 3.2. We consider $D_4^{(1)}$ and states from $B_6 \otimes B_3 \otimes B_4 \otimes B_4 \otimes B_2^{\otimes 8}$.

$$\begin{array}{cccccccccccc}
124\overline{321} \cdot & 234 \cdot & \overline{2321} \cdot & 13\overline{44} \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \\
111111 \cdot & 123 \cdot & 12\overline{32} \cdot & 34\overline{41} \cdot & \overline{32} \cdot & \overline{43} \cdot & \overline{34} \cdot & 12 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1123 \cdot & 1113 \cdot & \overline{33} \cdot & 44 \cdot & 23 \cdot & \overline{12} \cdot & \overline{32} \cdot & \overline{43} \cdot & \overline{34} \cdot & 12 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1123 \cdot & 13 \cdot & 11 \cdot & 11 \cdot & \overline{33} \cdot & 44 \cdot & 23 \cdot & 11 \cdot & \overline{12} \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 23 \cdot & 13 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & \overline{33} \cdot & 44 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11 \cdot & 11 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11 \\
111111 \cdot & 111 \cdot & 1111 \cdot & 1111 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 11 \cdot & 23 \cdot & 13 \cdot & 11
\end{array}$$

Here, \cdot represents \otimes .

Remark 3.3. Suppose $p_j = u_{l_j}$ for $1 \leq j \leq k$ in a state $p = p_1 \otimes \cdots \otimes p_L$. Then in the state $T_\infty(p) = p'_1 \otimes \cdots \otimes p'_L$, $p'_j = u_{l_j}$ is valid for $1 \leq j \leq k+1$.

We postpone the inverse scattering formalism of the dynamics to Theorem 5.2.

3.2. Counting particles and anti-particles. Recall that $\mathfrak{a}(\zeta)$ is defined in (2.24). In our present context, it is the number of all the particles and anti-particles within a box specified by $\zeta \in B_l$, where the term $2\overline{\zeta}_1$ means that a bound pair is regarded as a pair of a particle and an anti-particle (whose color is unspecified). The symbol \mathfrak{a} means **all** kinds of (anti-)particles.

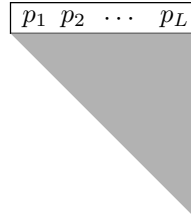
Let $p = p_1 \otimes \cdots \otimes p_L$ be a state and write its time evolution as

$$T_\infty^t(p_1 \otimes \cdots \otimes p_L) = p_1^t \otimes \cdots \otimes p_L^t,$$

where $p_j^t \in B_{l_j}$. We write $p_j = p_j^0 = (\zeta_{j,1}, \dots, \zeta_{j,n}, \overline{\zeta}_{j,n}, \dots, \overline{\zeta}_{j,1}) \in B_{l_j}$. For any elements a_1, \dots, a_r of $\{2, 3, \dots, n, \overline{n}, \dots, \overline{2}, \overline{1}\}$, we define the *counting function*

$$(3.2) \quad \rho_{a_1, \dots, a_r}(p) = \sum_{j=1}^L (\zeta_{j,a_1} + \cdots + \zeta_{j,a_r}) + \sum_{t \geq 1} \sum_{j=1}^L \mathfrak{a}(p_j^t),$$

where $\zeta_{j,\overline{3}} = \overline{\zeta}_{j,3}$, etc. The dependence on a_1, \dots, a_r enters the first term only. The indices in ρ_{a_1, \dots, a_r} will always be arranged in the order $2, 3, \dots, n, \overline{n}, \dots, \overline{3}, \overline{2}, \overline{1}$. The second term is finite due to Remark 3.3. In fact the double sum may well be restricted to $\sum_{t=1}^{L-1} \sum_{j=t+1}^L$ where the nonzero contributions are contained. This region is depicted as the SW quadrant of the time evolution patterns like Example 3.1 and 3.2.



(3.3)

The first term in (3.2) is the number of (anti-)particles with colors a_1, \dots, a_r contained in the top row which is the state p itself. The second term counts all kinds of particles and anti-particles in the hatched domain in (3.3).² By the definition it follows that

$$(3.4) \quad \rho_\emptyset(p) = \rho_{2, \dots, n, \overline{n}, \dots, \overline{2}, \overline{1}, \overline{1}}(T_\infty(p)).$$

² More precisely, it should be hatched in a staircase shape.

Given a state $p = p_1 \otimes \cdots \otimes p_L$, we write

$$(3.5) \quad p_{[k]} = p_1 \otimes \cdots \otimes p_k \quad (1 \leq k \leq L).$$

Example 3.4. Let p be the state in the first line in Example 3.2. Then, the counting function $\rho_{a_1, \dots, a_r}(p_{[k]})$ for $1 \leq k \leq 9$ takes the following values. (The middle column shows g such that $\rho_{a_1, \dots, a_r} = \rho_g$ in (3.7)–(3.10).)

a_1, \dots, a_r	g	$k = 1$	2	3	4	5	6	7	8	9
$2344\overline{3211}$	v_0	6	11	21	32	39	46	53	60	67
$2344\overline{321}$	v_1	5	10	19	30	37	44	51	58	65
$2344\overline{31}$	v_2	4	9	17	28	35	42	49	56	63
$234\overline{1}$	v_3	3	8	15	24	31	38	45	52	59
$23\overline{41}$	v_3^*	2	6	13	24	31	38	45	52	59
$2\overline{1}$	$w_2 - v_2$	2	5	12	20	27	34	41	48	55
$\overline{1}$	$w_1 - v_1$	1	3	9	17	24	31	38	45	52
\emptyset	$v_0^{\sigma_1}$	0	2	7	15	22	29	36	43	50

We set $\rho_{a_1, \dots, a_r}(p_{[0]}) = 0$ for any a_1, \dots, a_r . Consider the difference $\rho_{2\overline{1}}(p_{[k]}) - \rho_{\overline{1}}(p_{[k]})$ for example. By the definition (3.2), it is the number of color 2 particles contained in $p_{[k]}$. Thus we have

$$\begin{aligned} & \#(\text{color 2 particles}) \text{ in } p_k (= \zeta_{k,2}) \\ &= \rho_{2\overline{1}}(p_{[k]}) - \rho_{\overline{1}}(p_{[k]}) - \rho_{2\overline{1}}(p_{[k-1]}) + \rho_{\overline{1}}(p_{[k-1]}). \end{aligned}$$

This is an example of the relations that reproduces a local variable from non-local counting functions. Given l_k , the set of counting functions that are necessary and sufficient to completely reproduce the local state $p_k \in B_{l_k}$ is not unique. However there is a choice that is linked with the generalized energies in Section 2.3. By using the function γ_g in (2.25)–(2.29), we set

$$(3.6) \quad \rho_g(p) = \sum_{j=1}^L \gamma_g(p_j) + \sum_{t \geq 1} \sum_{j=1}^L \mathfrak{a}(p_j^t)$$

for $g = v_a$ ($0 \leq a \leq n-1$), $w_a - v_a$ ($1 \leq a \leq n-2$), v_{n-1}^* and $v_0^{\sigma_1}$. Although the notations ρ_g here and ρ_{a_1, \dots, a_r} in (3.2) are somewhat confusing, we dare to use the both in the sequel supposing the resemblance is not too serious. Then (3.6) is explicitly given as follows:

$$(3.7) \quad \rho_{v_a}(p) = \rho_{2, \dots, n, \overline{n}, \dots, \overline{a+1}, \overline{1}}(p) \quad (0 \leq a \leq n-2),$$

$$(3.8) \quad \rho_{v_{n-1}}(p) = \rho_{2, 3, \dots, n-1, n, \overline{1}}(p), \quad \rho_{v_{n-1}^*}(p) = \rho_{2, 3, \dots, n-1, \overline{n}, \overline{1}}(p),$$

$$(3.9) \quad \rho_{w_a - v_a}(p) = \rho_{2, 3, \dots, a, \overline{1}}(p) \quad (1 \leq a \leq n-2),$$

$$(3.10) \quad \rho_{v_0^{\sigma_1}}(p) = \rho_{\emptyset}(p).$$

The last one is subsidiary in that $\rho_{v_0^{\sigma_1}}(p) = \rho_{w_1 - v_1}(p) - \rho_{v_0}(p) + \rho_{v_1}(p)$ holds reflecting (2.12). One may also additionally introduce

$$\rho_{w_a}(p) = \rho_{w_a - v_a}(p) + \rho_{v_a}(p) = \sum_{j=1}^L \gamma_{w_a}(p_j) + 2 \sum_{t \geq 1} \sum_{j=1}^L \mathfrak{a}(p_j^t)$$

for $1 \leq a \leq n-2$. See (2.30). For $D_4^{(1)}$, the counting functions (3.7)–(3.10) are precisely those listed in Example 3.4.

Theorem 3.5. For $p_{[k]}$ in (3.5), set $\delta\rho_g = \rho_g(p_{[k]}) - \rho_g(p_{[k-1]})$. The counting functions (3.7)–(3.9) reproduce the local state $p_k = (\zeta_1, \dots, \zeta_n, \bar{\zeta}_n, \dots, \bar{\zeta}_1) \in B_{l_k}$ by

$$\begin{aligned}\zeta_1 &= l_k - \delta\rho_{v_0} + \delta\rho_{w_1-v_1}, \\ \zeta_a &= \delta\rho_{w_a-v_a} - \delta\rho_{w_{a-1}-v_{a-1}} \quad (2 \leq a \leq n-2), \\ \zeta_{n-1} &= \min(\delta\rho_{v_{n-1}}, \delta\rho_{v_{n-1}^*}) - \delta\rho_{w_{n-2}-v_{n-2}}, \\ \zeta_n &= \max(\delta\rho_{v_{n-1}} - \delta\rho_{v_{n-1}^*}, 0), \\ \bar{\zeta}_n &= \max(\delta\rho_{v_{n-1}^*} - \delta\rho_{v_{n-1}}, 0), \\ \bar{\zeta}_{n-1} &= -\max(\delta\rho_{v_{n-1}}, \delta\rho_{v_{n-1}^*}) + \delta\rho_{v_{n-2}}, \\ \bar{\zeta}_a &= \delta\rho_{v_{a-1}} - \delta\rho_{v_a} \quad (1 \leq a \leq n-2).\end{aligned}$$

Proof. Straightforward by using (3.2), (3.7)–(3.9) and $\min(\zeta_n, \bar{\zeta}_n) = 0$. \square

4. MAIN RESULT

4.1. Counting functions and generalized energies.

Theorem 4.1. For any state $p \in B_{l_1} \otimes \dots \otimes B_{l_L}$, the counting functions and the generalized energies (2.32) coincide, namely,

$$(4.1) \quad \mathcal{E}_g(p) = \rho_g(p)$$

for $g = v_a$ ($0 \leq a \leq n-1$), $w_a - v_a$ ($1 \leq a \leq n-2$), v_{n-1}^* and $v_0^{\sigma_1}$.

Here, $\mathcal{E}_{w_a-v_a}(p)$ should be understood as $\mathcal{E}_{w_a}(p) - \mathcal{E}_{v_a}(p)$, and the same convention is assumed for $I_{w_a-v_a}(p)$ in the sequel. Of course $\mathcal{E}_{w_a}(p) = \rho_{w_a}(p)$ follows as a corollary. The g 's in Theorem 4.1 are the same as those considered in Lemma 2.4 (ii) and (3.6). By substituting (3.7)–(3.10) into (4.1), the theorem may be rephrased as

$$\begin{aligned}\mathcal{E}_{v_a}(p) &= \rho_{2, \dots, n, \bar{n}, \dots, \overline{a+1}, \bar{1}}(p) \quad (0 \leq a \leq n-2), \\ \mathcal{E}_{v_{n-1}}(p) &= \rho_{2, 3, \dots, n-1, n, \bar{1}}(p), \quad \mathcal{E}_{v_{n-1}^*}(p) = \rho_{2, 3, \dots, n-1, \bar{n}, \bar{1}}(p), \\ \mathcal{E}_{w_a-v_a}(p) &= \rho_{2, 3, \dots, a, \bar{1}}(p) \quad (1 \leq a \leq n-2), \\ \mathcal{E}_{v_0^{\sigma_1}}(p) &= \rho_{\emptyset}(p).\end{aligned}$$

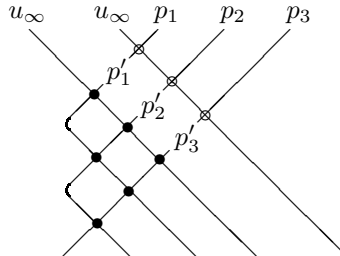
For the proof we need one more Lemma.

Lemma 4.2. Let $p = p_1 \otimes \dots \otimes p_L \in B_{l_1} \otimes \dots \otimes B_{l_L}$. For those g 's in Theorem 4.1, the following equality is valid:

$$(4.2) \quad \mathcal{E}_g(p) - \mathcal{E}_g(T_\infty(p)) = \sum_{i=1}^L g(\xi^{(i)} \otimes p_i),$$

where $\xi^{(i)} \in B_\infty$ is defined by $u_\infty \otimes p_1 \otimes \dots \otimes p_{i-1} \simeq p'_1 \otimes \dots \otimes p'_{i-1} \otimes \xi^{(i)}$.

Proof. The following proof simplifies the one for Proposition 4.6 in Ref. [17] in that the assumption $l_1 \geq \dots \geq l_L$ is not needed. We illustrate it for $L = 3$. Set $p = p_1 \otimes p_2 \otimes p_3$ and $T_\infty(p) = p'_1 \otimes p'_2 \otimes p'_3$. Then, Proposition 2.6 tells that $\mathcal{E}_g(T_\infty(p))$ is the sum of g at all \bullet in the following diagram.



On the other hand, the right hand side of (4.2) is equal to the sum of g at all \circ . Thus from Lemma 2.4 (ii), we find

$$\begin{aligned} \mathcal{E}_g(T_\infty(p)) + \sum_{i=1}^L g(\xi^{(i)} \otimes p_i) &= \sum_{1 \leq j \leq 3} I_g(u_\infty \otimes u_\infty \otimes p_1 \otimes \cdots \otimes p_j) \\ &\stackrel{(2.36)}{=} \sum_{1 \leq j \leq 3} I_g(u_\infty \otimes p_1 \otimes \cdots \otimes p_j) \stackrel{(2.35)}{=} \mathcal{E}_g(p), \end{aligned}$$

completing the proof. \square

Proof of Theorem 4.1. From Remark 3.3, $T_\infty^t(p) = u_{l_1} \otimes \cdots \otimes u_{l_L}$ for $t \geq L$. For such a state $\mathcal{E}_g = 0$ and $\rho_g = 0$ hold due to (2.23) and (3.7)–(3.10), respectively. Thus it suffices to show

$$\mathcal{E}_g(p) - \mathcal{E}_g(T_\infty(p)) = \rho_g(p) - \rho_g(T_\infty(p)).$$

By applying (4.2) and (3.6) to the left and the right hand sides, respectively, this becomes

$$\sum_{i=1}^L g(\xi^{(i)} \otimes p_i) = \sum_{i=1}^L (\gamma_g(p_i) + \mathfrak{a}(p'_i) - \gamma_g(p'_i)),$$

where we have set $T_\infty(p) = p'_1 \otimes \cdots \otimes p'_L$. (This was denoted by $p_1^1 \otimes \cdots \otimes p_L^1$ in (3.6).) From the definition of $\xi^{(i)}$ in (4.2), we have $\xi^{(i)} \otimes p_i \simeq p'_i \otimes \xi^{(i+1)}$. Therefore Lemma 2.2 tells that $g(\xi^{(i)} \otimes p_i) = \gamma_g(p_i) + \mathfrak{a}(p'_i) - \gamma_g(p'_i)$ holds for each i , finishing the proof. \square

4.2. *-transformed correspondence. Let us give an analogous result on $g = v_a^*$ ($1 \leq a \leq n-2$) which is not included in Theorem 4.1. Our presentation in this subsection is brief since the essential features are the same as the previous case. To state the result, let us introduce a *-transformed generalized energy and a *-transformed $D_n^{(1)}$ cellular automaton. (See (2.9) and (2.10) for the original definition of *.)

Let $u_l^* = (0, \dots, 0, l) \in B_l$. See (2.17). The *-transformed generalized energy $\mathcal{E}_{v_a^*}^*(p_L \otimes \cdots \otimes p_1)$ of an element $p_L \otimes \cdots \otimes p_1 \in B_{l_L} \otimes \cdots \otimes B_{l_1}$ is the sum of v_a^* for all the vertices in the following diagram ($L = 3$ example):

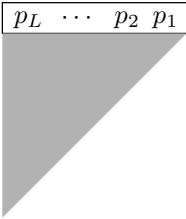
$$\mathcal{E}_{v_a^*}^*(p_3 \otimes p_2 \otimes p_1) =$$

Compare this with Proposition 2.6. One can show that $\mathcal{E}_{v_a^*}^*$ is well defined and R -invariant.

The *-transformed $D_n^{(1)}$ cellular automaton is the dynamical system on $B_{l_L} \otimes \cdots \otimes B_{l_1}$ endowed with the commuting time evolutions $T_l^*(l \geq 1)$ defined by $p_L \otimes \cdots \otimes p_1 \otimes u_l^* \simeq \xi \otimes T_l^*(p_L \otimes \cdots \otimes p_1)$. ($\xi \in B_l$ is determined by this relation.) T_∞^* is well defined. Moreover, under the time evolution $p'_L \otimes \cdots \otimes p'_1 = T_\infty^*(p_L \otimes \cdots \otimes p_1)$, the equality $p'_j = u_{l_j}^*$ is valid for $1 \leq j \leq k+1$ if $p_j = u_{l_j}^*$ for $1 \leq j \leq k$, which is parallel with Remark 3.3. For $\zeta \in B_l$, introduce the charge conjugation of (2.24)–(2.25) by

$$\begin{aligned} \mathfrak{a}^*(\zeta) &:= \mathfrak{a}(\zeta^*) = 2\zeta_1 + \zeta_2 + \cdots + \zeta_n + \bar{\zeta}_n + \cdots + \bar{\zeta}_2, \\ \gamma_{v_a^*}^*(\zeta) &:= \gamma_{v_a}(\zeta^*) = \zeta_1 + \zeta_{a+1} + \cdots + \zeta_n + \bar{\zeta}_n + \cdots + \bar{\zeta}_2 \quad (1 \leq a \leq n-2). \end{aligned}$$

Writing the time evolutions of a state $p = p_L \otimes \cdots \otimes p_1 \in B_{l_L} \otimes \cdots \otimes B_{l_1}$ as $(T_\infty^*)^t(p) = p_L^t \otimes \cdots \otimes p_1^t$, we define the counting function ($1 \leq a \leq n-2$):

$$\rho_{v_a^*}^*(p) = \sum_{j=1}^L \gamma_{v_a^*}^*(p_j) + \sum_{t \geq 1} \sum_{j=1}^L \mathfrak{a}^*(p_j^t) =$$


The counting is done by $\gamma_{v_a^*}^*$ for the top row and by \mathfrak{a}^* for the SE quadrant generated by T_∞^* beneath it. Thanks to the commutativity of $*$ and the combinatorial R (Prop.4.4 in Ref. [15]), Theorem 4.1 implies the following:

Proposition 4.3. *For any state $p \in B_{l_L} \otimes \cdots \otimes B_{l_1}$, the following equality is valid:*

$$\mathcal{E}_{v_a^*}^*(p) = \rho_{v_a^*}^*(p) \quad (1 \leq a \leq n-2).$$

This completes our interpretation of the $(*)$ -transformed) generalized energies associated with all the generalized local energies in terms of the $(*)$ -transformed) $D_n^{(1)}$ cellular automaton.

5. CONNECTION WITH COMBINATORIAL BETHE ANSATZ

Combinatorial Bethe ansatz was initiated by Kerov-Kirillov-Reshetikhin (KKR) [12, 13] to establish a fermionic formula of the Kostka-Foulkes polynomials with the invention of rigged configurations and the KKR bijection. Their fermionic formula generalized Bethe's formula [4] for some simplest Kostka numbers that originates in the completeness issue.

Rigged configurations are combinatorial analogue of solutions to the Bethe equations. The KKR bijection maps them to the combinatorial analogue of Bethe vectors which may be viewed as elements of (a subset of) $B_{l_1} \otimes \cdots \otimes B_{l_L}$. The combinatorial Bethe ansatz has flourished in the fermionic formulas for general affine Lie algebras [5, 19, 20, 21], the solution of the initial value problem of integrable $A_n^{(1)}$ cellular automata by the inverse scattering method [14] and a connection with the classical soliton theory [8] via ultradiscrete tau functions [17] and so forth. Our aim in this section is to present an inverse scattering formalism of the $D_n^{(1)}$ cellular automaton and to conjecture explicit formulas for some generalized energies in the form of ultradiscrete tau functions associated with the $D_n^{(1)}$ rigged configurations.

5.1. Inverse scattering formalism. Set $\mathcal{P}_+ = \{p \in B_{l_1} \otimes \cdots \otimes B_{l_L} \mid \tilde{e}_i p = 0 \text{ for } i = 1, 2, \dots, n\}$. A state belonging to \mathcal{P}_+ is called *highest*. It is known that there is a bijection between \mathcal{P}_+ and the set of *rigged configurations* [20, 21]. Consider a set

$$(5.1) \quad S = \{(a_i, j_i, r_i) \in \{1, 2, \dots, n\} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \mid i = 1, 2, \dots, N\},$$

where $N \geq 0$ is arbitrary and each triplet $s = (a, j, r)$ called *string* possesses color, length and rigging which will be denoted by $\text{cl}(s) = a$, $\text{lg}(s) = j$ and $\text{rg}(s) = r$, respectively.³ S is a rigged configuration if $\text{rg}(s) \leq p_{\text{lg}(s)}^{(\text{cl}(s))}$ is satisfied for all $s \in S$. Here $p_j^{(a)} = \delta_{a,1} \sum_{k=1}^L \min(j, l_k) - \sum_{t \in S} C_{a, \text{cl}(t)} \min(j, \text{lg}(t))$, where $(C_{a,b})_{1 \leq a, b \leq n}$ is the Cartan matrix of D_n . Note that $p_{\text{lg}(s)}^{(\text{cl}(s))} \geq 0$ has to be satisfied for all $s \in S$, which imposes a stringent condition on the set $\{(a_i, j_i) \mid i = 1, \dots, N\}$. Set $\text{RC} = \{S : \text{rigged configuration}\}$.

³Colors $1, 2, \dots, n$ of strings in rigged configurations should not be confused with colors $1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}$ of (anti)-particles.

Theorem 5.1 (Refs. [20] and [21]). *There is a bijection $\Phi : \mathcal{P}_+ \rightarrow \text{RC}$.*

An explicit algorithm to determine the image of $\Phi^{\pm 1}$ is known. It is a $D_n^{(1)}$ analogue of the Kerov-Kirillov-Reshetikhin bijection [13] for $A_n^{(1)}$, which plays a central role in the combinatorial Bethe ansatz. Our convention here is the one adopted in Ref. [14].

For any highest state $p \in \mathcal{P}_+$, its time evolution $T_l(p)$ is again highest. Thus T_l induces a time evolution on RC via Φ . Let L be large enough and assume the situation in (3.1). Then we have

Theorem 5.2. $\Phi(T_l(p)) = \tilde{T}_l \Phi(p)$ holds, where $\tilde{T}_l : \{(a_i, j_i, r_i)\} \mapsto \{(a_i, j_i, r_i + \delta_{a_i, 1} \min(j_i, l))\}$ is a linear flow on rigged configurations.

Proof. The proof uses Theorem 8.6 of Ref. [21] and is similar to that of Proposition 2.6 in Ref. [14] for type $A_n^{(1)}$. \square

Thus the composition $\Phi^{-1} \circ \tilde{T}_l \circ \Phi$ linearizes the original time evolution T_l and solves the initial value problem in the $D_n^{(1)}$ cellular automaton by the inverse scattering method. See Ref. [14] for an analogous result for $A_n^{(1)}$.

5.2. Conjecture on ultradiscrete tau functions. For a rigged configuration S (5.1), let $T \subseteq S$ be a (possibly empty) subset of S . In general, T is no longer a rigged configuration. We introduce the piecewise linear functions ($0 \leq k \leq L$ and $0 \leq d \leq n$)

$$\begin{aligned} c(T) &= \frac{1}{2} \sum_{s, t \in T} C_{\text{cl}(s), \text{cl}(t)} \min(\text{lg}(s), \text{lg}(t)) + \sum_{s \in T} \text{rg}(s), \\ c_k^{(d)}(T) &= c(T) - \sum_{i=1}^k \sum_{s \in T, \text{cl}(s)=1} \min(l_i, \text{lg}(s)) + \sum_{s \in T, \text{cl}(s)=d} \text{lg}(s) \end{aligned}$$

By the definition, the last term in $c_k^{(d)}(T)$ is 0 when $d = 0$, and the relation

$$(5.2) \quad c_k^{(d)}(T) = c_k^{(0)}(T)|_{\text{rg}(s) \rightarrow \text{rg}(s) + \text{lg}(s)\delta_{\text{cl}(s), d}}$$

holds. Obviously we have $c(\emptyset) = c_k^{(d)}(\emptyset) = 0$. On the other hand, $c(S)$ is known as the (co)charge of the rigged configuration S [13, 5, 20, 21].

We define a $\mathbb{Z}_{\geq 0}$ -valued piecewise linear function on S as follows:

$$(5.3) \quad \tau_k^{(d)}(S) = -\min_{T \subseteq S} \left(c_k^{(d)}(T) \right) \quad (0 \leq k \leq L, 0 \leq d \leq n).$$

For S in (5.1), the minimum extends over 2^N candidates and reminds us of the structure of tau functions in the theory of solitons [8]. In fact, for type $A_n^{(1)}$, analogous functions have been identified [17] as ultradiscretization of the tau functions in KP hierarchy. Although such an origin is yet to be clarified, we call (5.3) *ultradiscrete tau function*. Guided by the results in $A_n^{(1)}$ and supported by computer experiments, we propose

Conjecture 5.3. *For any highest state $p \in \mathcal{P}_+$, let $S = \Phi(p)$ be the corresponding rigged configuration. Then, the following equalities hold for $p_{[k]}$ (3.5) with $0 \leq k \leq L$.*

$$(5.4) \quad \tau_k^{(0)}(S) = \mathcal{E}_{v_0}(p_{[k]}),$$

$$(5.5) \quad \tau_k^{(1)}(S) = \mathcal{E}_{v_0^{\sigma_1}}(p_{[k]}),$$

$$(5.6) \quad \tau_k^{(n-1)}(S) = \mathcal{E}_{v_{n-1}^*}(p_{[k]}),$$

$$(5.7) \quad \tau_k^{(n)}(S) = \mathcal{E}_{v_{n-1}}(p_{[k]}),$$

$$(5.8) \quad \tau_k^{(2)}(S) = \mathcal{E}_{w_2}(p_{[k]}) - \mathcal{E}_{v_0}(p_{[k]}) + \varphi_0(p_{[k]}).$$

In (5.8), $\varphi_0(p_{[k]})$ is the standard notation in crystal theory meaning $\max\{j \geq 0 \mid \tilde{f}_0^j p_{[k]} \neq 0\}$. By using (3.4), (3.7) with $a = 0$, (3.10), (5.2) and Theorem 5.2, one can show that (5.4) and (5.5) are equivalent.

The algorithm [20, 21] for $\Phi^{\pm 1}$ seems valid not only for highest but *arbitrary* states if one allows negative rigging. With such a generalization, we expect that Theorem 5.2 and Conjecture 5.3 hold for any state, which was indeed the case for type $A_n^{(1)}$ [17].

Example 5.4. For the initial state p in Example 3.2, its rigged configuration is $S = \Phi(p) = \{(1, 8, -2), (1, 6, 0), (1, 2, -1), (1, 1, -1), (2, 8, 0), (2, 6, -1), (2, 2, -1), (3, 8, -3), (4, 8, -1)\}$. The ultradiscrete tau function $\tau_k^{(d)}(S)$ and $\varphi_0(p_{[k]})$ take the following values.

	$k = 1$	2	3	4	5	6	7	8	9
$\tau_k^{(0)}(S)$	6	11	21	32	39	46	53	60	67
$\tau_k^{(1)}(S)$	0	2	7	15	22	29	36	43	50
$\tau_k^{(2)}(S)$	1	3	9	16	23	30	37	44	51
$\tau_k^{(3)}(S)$	2	6	13	24	31	38	45	52	59
$\tau_k^{(4)}(S)$	3	8	15	24	31	38	45	52	59
$\varphi_0(p_{[k]})$	1	0	1	0	0	0	0	0	0

Comparing this with Example 3.4, one can check Conjecture 5.3.

Still many generalized energies in previous sections await formulas as in Conjecture 5.3 to be discovered. They are ultradiscrete analogue of the so called $X = M$ conjecture [5, 19] in the sense that the generalized energies from crystal theory acquire explicit formulas of a fermionic nature originating in the combinatorial Bethe ansatz. Such results combined with Theorems 3.5 and 4.1 will lead to a piecewise linear formula for the bijection Φ^{-1} as was done for $A_n^{(1)}$ [17].

Let us end by raising a closely related question as another future problem. Let $\mathcal{P}_+(\lambda)$ denote the subset of \mathcal{P}_+ having the prescribed weight λ . Then, does the generating function

$$X_g(\lambda) = \sum_{p \in \mathcal{P}_+(\lambda)} q^{\mathcal{E}_g(p)}$$

of the generalized energy admit a fermionic formula like Refs. [13] and [5]?

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